

# Learning Mathematics in Secondary School through Technology Enabled Explorations: A Perspective from India

*Jonaki B Ghosh*

e-mail: [jonakibghosh@gmail.com](mailto:jonakibghosh@gmail.com)

Lady Shri Ram College for Women

University of Delhi

New Delhi, INDIA

## Abstract

*This paper highlights the role of technology in mathematics education with examples from the Indian context. Mathematics forms the foundation of education in Indian society and is a compulsory subject up to grade 10. The last few years have witnessed extensive use of computer technology in school education. However in the vast majority of Indian classrooms, mathematics teaching continues in the traditional “chalk and board” manner. Technology, if used for teaching mathematics, is primarily for demonstration purposes and does not involve the student actively. Thus mathematics teaching, especially at the secondary and higher secondary school level, offers little opportunity for visualization and exploration. This renders mathematics as an abstract subject as seen from the eyes of a majority of students. The National Curriculum Framework of India (NCF, 2005) in its position paper Teaching of Mathematics recommends the use of technology in the form of computer software and calculators, “Technology can greatly aid the process of mathematical explorations,....If ordinary calculators can offer such possibilities, the potential of graphing calculators and computers for mathematical exploration is far higher”. The recommendations of the NCF, has been a motivating factor for conducting studies related to the use of technology in the mathematics classroom. In this paper, we shall describe a study conducted with higher secondary school students, where technology in the form of graphics calculators, spreadsheets and Computer Algebra Systems played a key role in developing their mathematical thinking. We will illustrate that by enabling students to explore mathematical applications through technology, a rich and motivating environment may be created in the classroom for exploring mathematical ideas.*

## 1. Introduction

The last few decades have witnessed a major shift of paradigm as far as mathematics teaching and learning is concerned, in many countries across the world. Mathematics education is being revolutionized with the advent of new and powerful technological tools in the form of dynamic geometry software (DGS), computer algebra systems (CAS), spreadsheets and graphic calculators which enable students to focus on exploring, conjecturing, reasoning and problem solving and not be weighed down by rote memorization of procedures, computational algorithms, paper-pencil-drills and symbol manipulation. Some research studies have shown that the appropriate use of technology can lead to better conceptual understanding and make higher level processes accessible to students without compromising on paper – pencil skills [11]. One of the first leading studies in the use of technology in mathematics education is the one reported by Heid [6] in which computer algebra and some graphing tools were integrated into a first year university calculus course. Technology provided scope for a “concept first” approach, in which, calculus concepts were extensively taught using the tools and computational aspects were dealt with at the end of the course. The results showed that technology helped to develop conceptual understanding without hampering procedural skills. In another study, Heid [7] describes CAS as a cognitive technology which makes higher level mathematical processes accessible to students. Kissane [8] describes some “pedagogical affordances” of the graphics calculator in visualizing and exploring concepts in calculus. According to him, “procedural competence can be best developed when students understand the underlying ideas well” (p. 16). Laborde [9] suggests that learning geometry with a Dynamic Geometry Environment (DGE) might offer students possibilities to construct and

manipulate geometrical figures and do empirical investigations. Leung [10] describes that “a key feature of DGE is its ability to visually represent geometrical invariants amidst simultaneous variations induced by dragging activities” (p. 2). Thus mathematics education researchers across the world have conducted many studies which support the use of technology in mathematics teaching and learning.

While many countries are increasingly using technology for mathematics instruction, in the vast majority of schools in India, mathematics continues to be taught in the traditional 'chalk and board' manner. Also very few research studies have been conducted to study the impact of integration of technology for mathematics teaching and learning. In this paper, we shall highlight the opportunities for using technology in mathematics classrooms in India by describing a research study where various modules based applications of mathematics were conducted with senior secondary school students. The study aimed to explore how learning of mathematical concepts is impacted by integrating mathematical modelling into the senior secondary mathematics curriculum. The objective was to reflect upon the following research questions

- (i) How does the integration of applications and modelling activities impact student's understanding of mathematical concepts?
- (ii) Does introducing mathematical modelling activities, enabled by technology, help students access higher level mathematical concepts?

The paper is divided into four sections. In section 2 we create a context for the study by highlighting some aspects of the senior secondary mathematics curriculum in India. In section 3, we describe three modules in which students of grades 11 and 12 used spreadsheets (MS Excel) and CAS (Mathematica) to explore applications of topics taught in the curriculum. We shall illustrate that the integration of technology enabled explorations of mathematical applications through these modules, aided by facilitation from the teacher, provided students with a rich and motivating environment, for exploring important mathematical concepts. Further, these modules helped students to appreciate the relevance of the mathematics they learnt in school and also in learning new mathematical content.

## **2. The Senior Secondary Mathematics Curriculum in India**

The senior secondary stage is a significant landmark in the life of a student. In fact the National Curriculum Framework (NCF) 2005 [12] in its position paper on the ‘Teaching of Mathematics’ describes the higher secondary stage as the “launching pad from which the student is guided towards career choices.” At this stage the student has to make a choice between the science, commerce or humanities stream and has to appear for the school leaving examination (known as the board examination) at the end of grade 12. This is a high stakes examination as her performance in this examination holds the key to her future in terms of career opportunities. Thus mathematics teaching in grades 11 and 12 is largely driven by preparation for the board examination. Topics are taught in the traditional ‘chalk and board’ manner, without any substantial reference to their applications to real life problems. The mastery of procedural knowledge is emphasised leaving very little opportunity for visualization, exploration or discovery. Though the last few years have witnessed extensive use of computer technology in school education, there is hardly any use of technology for teaching and learning of mathematics. Technology, if used for teaching mathematics, is primarily for demonstration purposes and does not involve the student actively.

The position paper on teaching of mathematics, recommends that mathematics teaching at all levels should be made more ‘activity oriented’. While this has been the primary driving force for revisiting and revamping the elementary school mathematics curriculum, it has had little impact on the senior secondary curriculum. At this level, the textbooks, which happen to be the only resource for teachers and students, are usually examination oriented and mathematics teaching is very

pedantic. Topics such as Sets, Relations, Logic, Sequences and series, Linear inequalities, Combinatorics, Conic sections and Complex numbers are introduced in grade 11. The curriculum in grade 12 is dominated by differential and integral calculus accounting for almost half of the content. Other topics in grade 12 include Matrices and Determinants, Vector Algebra, Three Dimensional Geometry, Linear Programming and Probability. These topics are required to be studied by all students irrespective of the stream chosen by them, be it Humanities, Commerce or the basic sciences. Further the board examinations will test the manipulative and computational aspects of these topics, rather than visualization and exploration of concepts and ideas. To highlight this point here are two sample questions from the topic of Differentiation in the text book of grade 12.

1. If  $y = (\tan^{-1} x)^2$  then show that  $(x^2+1)^2 \frac{d^2 y}{dx^2} + 2x(x^2+1) \frac{dy}{dx} = 2$

2. Differentiate  $\log \sqrt{\frac{1+\cos^2 x}{1-e^{2x}}}$  with respect to  $x$ .

It may be worth pointing out here that the mathematical content knowledge of students at the senior secondary level is uniform across schools and regions. In the traditional classroom, the emphasis is on developing procedural knowledge through drill and practice and on learning ‘tricks’ to solve problems. Getting exact answers is often over emphasised and the NCF 2005 recommends that we need to ‘liberate school mathematics from the tyranny of the one right answer, found by applying the one algorithm taught.’ It strongly recommends the inclusion of modelling activities in the curriculum and suggests that this can be done in the form of investigatory exercises or projects which will enable the student to see the relevance of the mathematics taught at school. Another important recommendation of the NCF 2005 is that of making a shift from ‘content to processes’. It states that “content areas of mathematics addressed in our schools do offer a solid foundation.....What can be levelled as major criticism against our extant curriculum and pedagogy is its failure with regard to mathematical processes.....formal problem solving, use of heuristics, estimation and approximation, optimisation, use of patterns, visualisation, representation, reasoning and proof, making connections, mathematical communication.....Giving importance to these processes constitutes the difference between doing mathematics and swallowing mathematics, between mathematisation of thinking and memorising formulae, between trivial mathematics and important mathematics.....”. In order to achieve this shift, especially at the senior secondary level, the NCF also advocates the use of technology in the form of computer software and calculators, “Technology can greatly aid the process of mathematical explorations,....If ordinary calculators can offer such possibilities, the potential of graphing calculators and computers for mathematical exploration is far higher”. The recommendations of the NCF, has been a motivating factor for conducting the research study related to the use of technology in the mathematics classroom which is described in this paper.

### 3. The Research Study: Modules on applications of mathematics

In this section we shall describe how learning is impacted by integrating mathematical modelling and applications into the senior secondary mathematics curriculum. 30 grade 12 students and 25 grade 11 students selected from three schools in New Delhi underwent modules where they explored a few topics of the curriculum through mathematical models and applications. Each module was spread over 5 to 6 sessions of approximately two hours each, where each session was conducted on a school day after school hours. MS Excel and Mathematica were used as the vehicles of exploration by students. A description of a similar study integrating the use of graphics calculators may be found in Ghosh [4]. Worksheets were provided where they had to record their paper – pencil work and write their findings and observations. At the end of the module they were

required to respond to a short questionnaire and give a written feedback describing their learning experience in the module.

### 3.1. Educational Setting

All students who were a part of the study were from three Delhi schools which follow the curriculum prescribed by the Central Board of Secondary Education (CBSE), a national board for school education in India. The CBSE does not prescribe the use of technology for teaching mathematics or permit its use in examinations. Individual schools however have the freedom to integrate technology in their classrooms. The participants of the study were rated as above average or high performing students. Among them, grade 12 students explored applications of Matrices and Determinants and Calculus, while grade 11 students explored applications of Probability. Though they had studied these topics as a part of the curriculum in their regular classes, the emphasis had been largely on computational skills and mastery of methods or algorithms to solve problems. The mathematical content of these topics was presented without any reference to application of the concepts in solving practical problems. For example, the topic, Matrices and Determinants comprises of the subtopics on types of matrices, matrix operations, inverse of a matrix, computing determinants and their properties and solutions of systems of equations. The emphasis is purely on computational aspects of manipulating matrices and determinants. Some sample exercises from the the topics of Matrices, given in the textbook are as follows

1. If  $A = \begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix}$  and  $f(x) = 2x^3 + 4x + 5$ , find  $f(A)$ .

2. Using properties of determinants prove that

$$\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} = abc(a-b)(b-c)(c-a)$$

The author (who was also their teacher) decided to design modules which would emphasise the use of applications and mathematical models to introduce concepts and procedures. The thrust was on highlighting the relevance of the mathematical concepts as a tool for solving problems. The mathematical models were selected from varied topics such as Genetics, Forest management, Consumer behaviour, Fractals, Probabilistic games and Cryptography. In all the sessions of the modules, students were given worksheets which required them to explore a mathematical model and record their solutions or observations. They were given access to MS Excel and Mathematica for computational work. Prior to the module, a few workshop sessions were conducted to familiarize the students with performing mathematical computations on Mathematica and MS Excel. It may be worth commenting that although some of the mathematical content of the applications explored by students in these modules are beyond the prescribed curriculum, these can be made accessible to students by providing access to appropriate technology and through proper facilitation. In the rest of this paper we shall describe the different mathematical models which were explored by the participants of the study.

### 3.2. Module 1: The Hill Cipher Method

The grade 12 students explored the Hill cipher method [2], an application of matrices to cryptography. During the exploration, students were encouraged to do certain procedures, such as, finding the product and determinants of  $2 \times 2$  and  $3 \times 3$  matrices by hand. They were however recommended to

use technology only while working with higher order matrices. The objective was to maintain a balance between technology enabled explorations and procedural skills. At the beginning of the module, they were familiar with the concept of matrices and basic matrix algebra but could not cite any practical use of matrices. They described a matrix as a ‘rectangular arrangement of numbers in rows and columns’ and said that the utility of matrices had not been discussed in their regular class. This prompted the author to focus the first two sessions of the module on the use of matrices in modelling practical problems. Examples such as ranking teams in a tournament and modelling consumer purchasing behaviour using transition matrices were introduced. Matrix operations of addition, subtraction, scalar multiplication and multiplication were used to analyse the models in the third session. This created a context for using matrices in real life problems. In the fourth session the concept of inverse of a matrix was recapitulated. Students were aware of the methods for finding the inverse of  $2 \times 2$  and  $3 \times 3$  matrices. Many students were curious as to whether there is any practical use of the inverse of a matrix. At this point the author decided to introduce students to the Hill Cipher method. They were familiarised with some preliminary information related to ciphers.

Ciphers are methods for transforming a given message, the plaintext, into a new set of characters which is unintelligible to anyone who does not know the key (the transformation used to convert the plaintext), known as the ciphertext. The secret rule, that is, the inverse key, is required to reverse the transformation in order to recover the original message. After this, students were familiarised with the following preliminary definitions related to modular arithmetic (a topic which is not a part of the syllabus).

**Definition 1:** Given an integer  $m > 1$ , called the *modulus*, the two integers  $a$  and  $b$  are said to be *congruent* to one another *modulo*  $m$  if the difference  $a - b$  is an integral multiple of  $m$  or  $a = b + km$  for some integer  $k$  (positive, negative or zero).

**Definition 2:** Let  $m$  be an integer with  $m > 1$ . For an arbitrary integer  $a$ , the *residue of  $a$  modulo  $m$*  is the unique integer  $r$  among  $0, 1, 2, \dots, m - 1$  to which  $a$  is congruent modulo  $m$ .

Thus  $23 \equiv 7 \pmod{8}$ , since  $23 = 2 \times 8 + 7$ . Here  $a = 23$ ,  $m = 8$  and  $r = 7$ . Also  $-18 = -3 \times 8 + 6$ . Thus  $r = 6$  and we can write  $-18 \equiv 6 \pmod{8}$ .

Students were given a worksheet to familiarize them with computations related to the above definitions. After this they were then introduced to the Hill Cipher method and were given 29 characters along with their numerical values as shown in Table 1.

Table 1. A substitution table of 29 characters for the Hill Cipher Method

A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
P	Q	R	S	T	U	V	W	X	Y	Z	.	_	?	
15	16	17	18	19	20	21	22	23	24	25	26	27	28	

The encoding matrix or key was chosen as  $E = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}$ . Students were then introduced to the encoding process (converting the plaintext to the ciphertext) through the following steps.

Step 1. Let the plaintext message be **HOW\_R\_U?**

Step 2. Convert it to its substitution values from the substitution table and group them in pairs

7 14    22 27    17 27    20 28

Each pair will form a column of the message matrix M.

Step 3. Compute the product EM and reduce it modulo 29.

$$EM = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} 7 & 22 & 17 & 20 \\ 14 & 27 & 27 & 28 \end{bmatrix} = \begin{bmatrix} 35 & 76 & 71 & 76 \\ 119 & 255 & 240 & 256 \end{bmatrix} = \begin{bmatrix} 6 & 18 & 13 & 18 \\ 3 & 23 & 8 & 24 \end{bmatrix} \pmod{29}$$

Step 4. The pairs of numbers 6 3 18 23 13 8 18 24

lead to the encrypted message or ciphertext **GDSXNISY**.

Some students used Mathematica for the matrix computations while others used MMULT and MINVERSE commands in Excel. Figure 1 shows screenshots of the encoding process on Excel and figure 2 shows similar computations on Mathematica.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N
1														
2														
3		1	2		7	22	17	20		35	76	71	76	
4		3	7		14	27	27	28		119	255	240	256	
5														
6										6	18	13	18	
7										3	23	8	24	

Figure 1: an Excel screenshot of the encoding process as done by the students

To decode the cipher text the inverse of the matrix  $E = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}$ , that is  $E^{-1} = \begin{bmatrix} 7 & -2 \\ -3 & 1 \end{bmatrix}$  is

required. The decoding process was introduced as follows

Step 1. The ciphertext message reaches the receiver as **GDSXNISY**

Step 2. Convert the characters to their respective Hill-2-cipher values and make pairs.

6 3 18 23 13 8 18 24

Each pair will form a column of a matrix N.

Step 3. Compute the product  $E^{-1}N$  and reduce it modulo 29.

$$E^{-1}N = \begin{bmatrix} 7 & -2 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 6 & 18 & 13 & 18 \\ 3 & 23 & 8 & 24 \end{bmatrix} = \begin{bmatrix} 36 & 80 & 75 & 78 \\ -15 & -31 & -31 & -30 \end{bmatrix} = \begin{bmatrix} 7 & 22 & 17 & 20 \\ 14 & 27 & 27 & 28 \end{bmatrix} \pmod{29}$$

When the values are written column wise they become

7 14 22 27 17 27 20 28

which translates to **HOW\_R\_U?**

The above was illustrated as an example of a hill-2-cipher. By definition, an arbitrary *Hill n – cipher* has as its key a given invertible  $n \times n$  matrix whose entries are non – negative integers from among  $0,1,\dots, m - 1$ , where  $m$  is the number of characters used for the encoding process.

```

Wolfram Mathematica 7.0 - [hill.nb]
File Edit Insert Format Cell Graphics Evaluation Palettes Window Help

hill.nb

A =  $\begin{pmatrix} 1 & 2 \\ 3 & 7 \end{pmatrix}$ ;
M =  $\begin{pmatrix} 7 & 22 & 17 & 20 \\ 14 & 27 & 27 & 28 \end{pmatrix}$ ;
ciphertext = Mod[A.M, 29]
% // MatrixForm

{{6, 18, 13, 18}, {3, 23, 8, 24}}

 $\begin{pmatrix} 6 & 18 & 13 & 18 \\ 3 & 23 & 8 & 24 \end{pmatrix}$ 

Inverse[A].ciphertext // MatrixForm

 $\begin{pmatrix} 36 & 80 & 75 & 78 \\ -15 & -31 & -31 & -30 \end{pmatrix}$ 

plaintext = Mod[%, 29] // MatrixForm

 $\begin{pmatrix} 7 & 22 & 17 & 20 \\ 14 & 27 & 27 & 28 \end{pmatrix}$ 

```

Figure 2: Mathematica code used by students for Hill Cipher computations

### Students’ explorations

Students found this technique very exciting. They wanted to try out examples of Hill-3 –ciphers. For many it was a challenge to come up with 3 by 3 matrices with determinant 1. However a few students

came up with interesting examples such as  $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 2 & 9 \end{bmatrix}$  and  $\begin{bmatrix} 2 & 0 & -1 \\ 0 & 1 & 2 \\ -1 & 3 & 7 \end{bmatrix}$ . They used their

knowledge of determinants to create higher order matrices with determinant equal to 1.

Students were then divided into groups of three. Out of the ten groups, six were able to work with Hill-3-ciphers. Two groups ventured into Hill-4 –ciphers and the rest perfected their work with Hill-2-ciphers. After this the class designed a ‘cipherquiz’. Each group had to encode a message and post it on the whiteboard after declaring the key. The ‘receivers’ had to use the inverse key to decode the message. All students performed the computations using technology. By the end of the fifth session, all 30 students were able to work with Hill ciphers.

After this, students began to come up with queries – what if the determinant of the encoding matrix is not 1? Can anyone crack the Hill cipher? In fact one student asked whether it is possible for a person (other than receiver or sender) to crack the code, that is, figure out the encoding matrix if

the plaintext and ciphertext are known. This was undoubtedly the ‘high point’ of the module for the author!

To deal with the first question, the author asked students to encode a plaintext using a matrix with determinant other than 1.  $E = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$  (with determinant is equal to -2), was one of the selected matrices. Since  $E^{-1}$  has fractional entries, the decoding process posed a problem and did not give back the original plain text. The author took this opportunity to familiarize the students with the method of finding the inverse of a matrix in  $Z_{29}$ . This means finding a matrix  $\begin{bmatrix} x & y \\ z & w \end{bmatrix}$  so that  $\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x & y \\ z & w \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \pmod{29}$ , where x,y,z and w are integers from 0 to 28. Figure 3 shows the by hand computations as done by as student.

Handwritten student work showing the step-by-step process of finding the inverse of matrix  $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$  modulo 29. The work starts with the augmented matrix  $[A | I] = \begin{bmatrix} 2 & 3 & 1 & 0 \\ 4 & 5 & 0 & 1 \end{bmatrix}$  and proceeds through several row operations:  $R_1 \rightarrow 15R_1$ ,  $R_2 \rightarrow R_2 - 4R_1$ ,  $R_2 \rightarrow 28R_2$ , and  $R_1 \rightarrow R_1 - 16R_2$ , eventually arriving at  $[I | A^{-1}] = \begin{bmatrix} 1 & 16 & 12 & 16 \\ 0 & 1 & 2 & 28 \end{bmatrix}$ . The final result is  $A^{-1} \text{ in } Z_{29} \text{ is } \begin{bmatrix} 12 & 16 \\ 2 & 28 \end{bmatrix}$ .

Figure 3: finding the inverse of a matrix in the set of integers modulo 29

To answer the second query, students were introduced to the Cracking theorem. Suppose the length  $m$  of the alphabet is a prime. Let  $P = [\bar{p}_1 \ \bar{p}_2 \ \dots \ \bar{p}_n]$  and  $C = [c_1 \ c_2 \ \dots \ c_n]$  be the plaintext matrix and ciphertext matrices respectively. Then the elementary row transformations that reduce  $C^T$  to the identity matrix  $I$  also reduce the matrix  $P^T$  to  $(A^{-1})^T$ .

A few students took up the challenge of applying the theorem. Since they were familiar with elementary row transformations, they could understand the method with some facilitation from the author. As an example, the plaintext MATH and the corresponding ciphertext MYSO were used to check if the key (which was chosen as  $A = \begin{bmatrix} 1 & 4 \\ 2 & 9 \end{bmatrix}$  in this case) could be ‘cracked’. Figure 4 shows the work done by a student. It was really satisfying when some students took up the challenge of writing a C++ program for the Hill Cipher method. During this attempt, they used their knowledge of matrix theory and logical thinking to obtain the desired output.



Cracking theorem  
 $[C^T | P^T] \rightarrow [I | (A^{-1})^T]$   
 Plain text  $\rightarrow$  MATH  $\rightarrow$  12 0 19 7  
 Cipher text  $\rightarrow$  MYSO  $\rightarrow$  12 24 18 14  
 $P = \begin{bmatrix} 12 & 19 \\ 0 & 7 \end{bmatrix}$      $C = \begin{bmatrix} 12 & 18 \\ 24 & 14 \end{bmatrix}$   
 $C^T | P^T = \left[ \begin{array}{cc|cc} 12 & 24 & 12 & 0 \\ 18 & 14 & 19 & 7 \end{array} \right] \xrightarrow{R_1 \rightarrow 17R_1} \left[ \begin{array}{cc|cc} 204 & 408 & 204 & 0 \\ 18 & 14 & 19 & 7 \end{array} \right]$   
 $\equiv \left[ \begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 18 & 14 & 19 & 7 \end{array} \right] \xrightarrow{R_2 \rightarrow R_2 - 18R_1} \left[ \begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -22 & 1 & 7 \end{array} \right]$   
 $\equiv \left[ \begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 7 & 1 & 7 \end{array} \right] \xrightarrow{R_2 \rightarrow 25R_2} \left[ \begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 175 & 25 & 175 \end{array} \right]$   
 $\equiv \left[ \begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & 25 & 1 \end{array} \right] \xrightarrow{R_1 \rightarrow R_1 - 2R_2} \left[ \begin{array}{cc|cc} 1 & 0 & -49 & -2 \\ 0 & 1 & 25 & 1 \end{array} \right]$   
 $\equiv \left[ \begin{array}{cc|cc} 1 & 0 & 9 & 27 \\ 0 & 1 & 25 & 1 \end{array} \right] = [I | (A^{-1})^T]$   
 $\therefore A^{-1} = \begin{bmatrix} 9 & 25 \\ 27 & 1 \end{bmatrix}$  in  $Z_{29}$   
 To find  $A^{-1} \left[ \begin{array}{cc|cc} 9 & 25 & 1 & 0 \\ 27 & 1 & 0 & 1 \end{array} \right] \leftrightarrow \left[ \begin{array}{cc|cc} 1 & 0 & 1 & 4 \\ 0 & 1 & 2 & 9 \end{array} \right]$   
 $\downarrow$   
 $A^{-1}$

Figure 4: a student's work on applying the Cracking Theorem

### 3.3. Module 2: Exploring the Monty Hall problem

Students often find it hard to understand the concept of conditional probability which is introduced in grade 11 as a part of the topic of probability. However, most problems in the curriculum deal with random experiments related to coin tossing, dice throwing, pack of playing cards etc. To create interest in students, the author decided to introduce them (grade 11 students) to the Monty Hall Problem (also referred to as the three door problem) which goes as follows

*You are a contestant in a game show where the host (Monty) asks you to choose one among three doors. Behind one of the doors is a car (the prize) and behind the other two are goats. After you select a door, the host doesn't open it; instead, he opens one of the other two doors and reveals a goat (the host, who knows what's behind the doors, never reveals the car). The host then asks you whether you want to stick to your original choice or you would like to switch by choosing the other door. The question is which option is more likely to win you the prize?*

In the first session of this module, the author encouraged the students to analyse the game by playing it among themselves. Indeed this exercise led to an interesting classroom discussion. The immediate reaction of most students (like most people) was that switching cannot matter and that the probability of winning the car (after Monty opens one of the doors and reveals a goat) is 1/2. Most students seemed to be convinced about this and could not see the fallacy in the incorrect reasoning. The probability that the player will select the door with the car (and therefore win) is 1/3, and probability that he will lose is 2/3. If his initial choice was correct then switching would be wrong. However if he chose a wrong door initially (the probability of which is 2/3) then switching would lead him to win (since the host will reveal the goat behind the other wrong door). Figure 5 shows a tree diagram which was used by the author to explain the problem. Here it was assumed that the player has chosen door 1. The tree diagram shows that the total probability of winning by switching is 2/3 whereas the probability of winning by staying with the original choice is 1/3. Thus it is always wiser to switch.

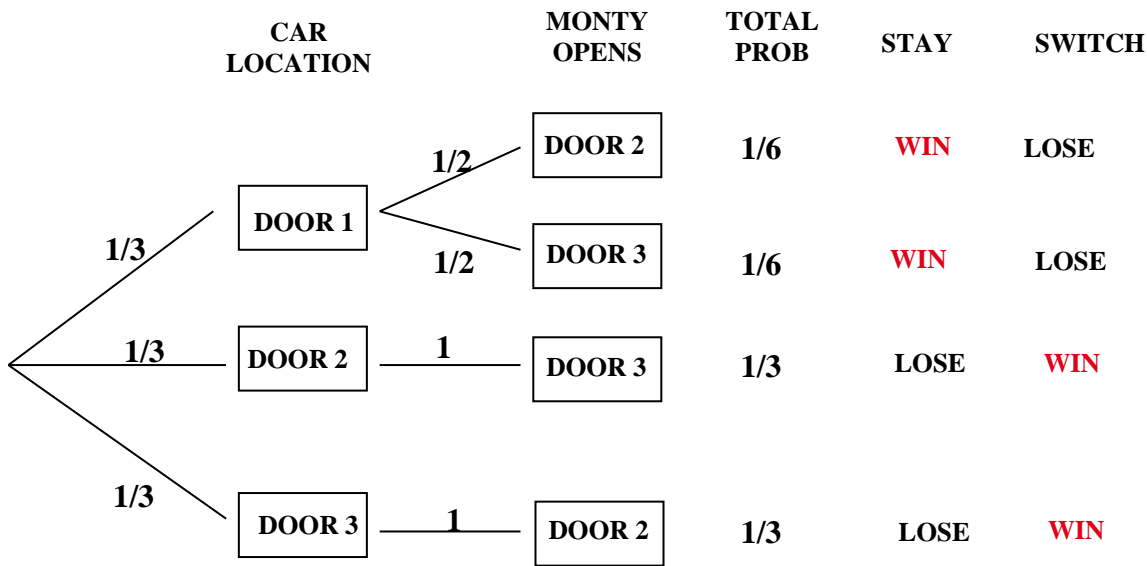


Figure 5: a tree diagram to explain the Monty Hall problem where the player's initial choice is door 1.

Students were then guided to verify the solution by simulating it in MS Excel using the RAND( ), INT( ) and IF( ) commands. In Figure 6, the experiment has been simulated 100 times with the assumption that the player's initial choice is door 1. The numbers in column A indicate the door behind which the car is placed. This is done by randomly generating the integers 1 through 3, both inclusive. We assume that the player's initial choice is door 1 in all the simulations and enter 1 throughout in column B. To get the entries in column C, we first check whether the entries for A and B are the same (i.e., the player has selected the door with the car). If so, we generate a random number between 0 and 1. If the entries of column A and B are both equal to 1, that is, the car is behind door 1 (which is also chosen by the player), then Monty may open either door 2 or door 3 (with probability 1/2 each) since by opening one of these doors he will not reveal the car. This is indicated by the random number between 0 and 1. A random number less than 0.5 indicates that Monty opens door 2 in which case 2 is entered in column C. However, if the random number is 0.5 or greater, then 3 is entered in column C indicating Monty's opening door 3.

These entries in column C are filled using a simple macro. In column D, the cases where the player wins by switching are identified. If the entries of columns A and B are different, then YES appears in the corresponding cell. Similarly in column E, the cases where the player wins by staying with his original choice are identified. Thus, if the entries of columns A and B are the same, then YES appears in the corresponding cell in column D. Finally counting the YES's in column D reveals the total number of times the player wins by switching in 100 trials. Interestingly, this number, on an average turns out to be around 67. Thus, by simulating the experiment in Excel, students were able to conclude that the probability of winning by staying with one's choice is around 33% and that by switching is 67%. Refer to [3] to see the steps of simulation used by the students in detail.

After going through the Excel simulation, the teacher emphasized the need to understand the problem using mathematical notation. The sample space of the experiment was written out in the form of triples (a,b,c) where a, b and c respectively denote the cars' position, the player's choice and the door opened by Monty (See Figure 7). It was interesting to see students' attempts at creating tree diagrams which led them to compute the probabilities of winning by switching (2/3) and sticking to one's choice (1/3). There was an element of surprise when they discovered that all the 12 outcomes of the sample space are not equally likely.

	A	B	C	D	E	F	G	H
1	CAR	PLAYER	MONTY	SWITCH	STAY		win by switch	win by staying
2	1	1	2	NO	YES		0.63	0.37
3	1	1	2	NO	YES			
4	1	1	3	NO	YES			
5	2	1	3	YES	NO			
6	2	1	3	YES	NO			
7	2	1	3	YES	NO			
8	3	1	2	YES	NO			
9	1	1	3	NO	YES			
10	2	1	3	YES	NO			
11	1	1	3	NO	YES			
12	3	1	2	YES	NO			
13	1	1	3	NO	YES			
14	1	1	2	NO	YES			
15	3	1	2	YES	NO			
16	2	1	3	YES	NO			

Figure 6: simulating the Monty Hall problem using Excel

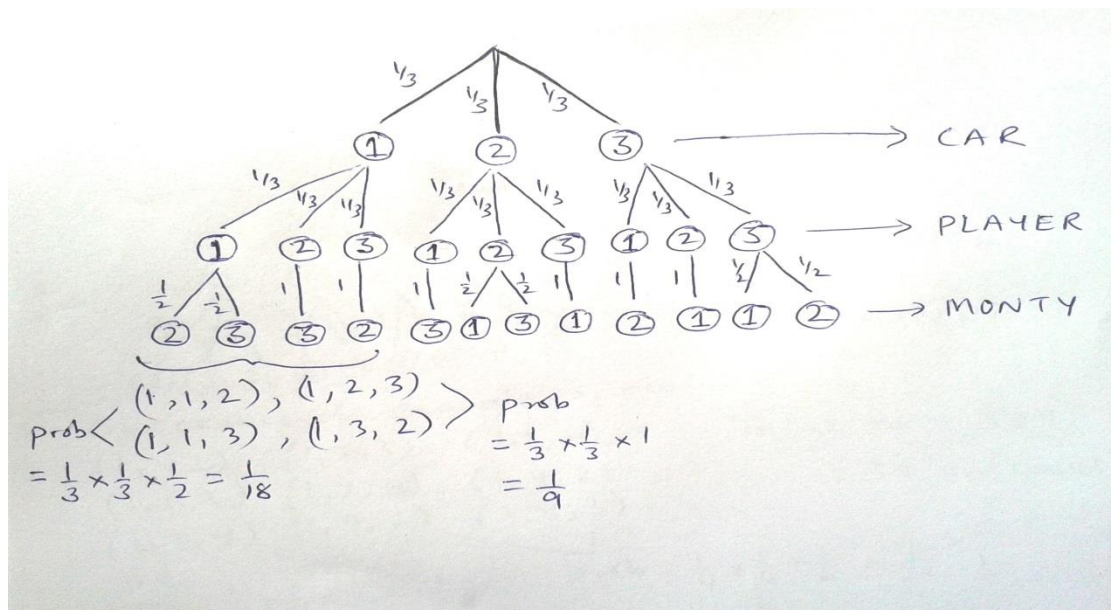


Figure 7: a tree diagram to represent the sample space of the Monty Hall problem (three door case).

The problem was further analysed using the notation of conditional probability (as this is also a requirement of the curriculum)

Let  $C_i$  denote the event that the car is behind door  $i$ . Clearly  $P(C_i) = \frac{1}{3}$  where  $i = 1, 2, 3$ .

Also let  $M_{ij}$  denote the event that Monty opens door  $j$  when the player chooses door  $i$ .

$$P(M_{ij}/C_k) = \begin{cases} 0, & \text{if } i = j \\ 0, & \text{if } j = k \\ 1/2, & \text{if } i = k \\ 1, & \text{if } i \neq k, j \neq k \end{cases} \quad 11$$

Here  $i$  denotes the door chosen by the player,  $j$  denotes the door opened by Monty and  $k$  denotes the door behind which the car is placed.

Now,  $P(M_{ij}/C_k) = 0$  if  $i = j$ , since Monty will never open the door chosen by the player. Also  $P(M_{ij}/C_k) = 0$  if  $j = k$ , since Monty will never open the door which has the car behind it. If the player chooses the door with the car behind it (that is,  $i = k$ ), then Monty can open any of the other two doors with probability  $\frac{1}{2}$  each. Hence  $P(M_{ij}/C_k) = \frac{1}{2}$  for  $i = k$ . For the case  $i \neq j \neq k$ , that is, the case when the player chooses a door which does not have the car behind it, Monty has no option but to open the third door (the one which was not chosen by the player and the one which does not have the car behind it). Hence  $P(M_{ij}/C_k) = 1$  for  $i \neq j \neq k$ . Using this notation, the teacher helped students to compute the probability of the event  $M_{13}$  (the event that Monty opens door 3 when the player chooses door 1).

$$\begin{aligned} P(M_{13}) &= P(M_{13} \cap C_1) + P(M_{13} \cap C_2) + P(M_{13} \cap C_3) \\ &= P(M_{13}/C_1)P(C_1) + P(M_{13}/C_2)P(C_2) + P(M_{13}/C_3)P(C_3) \\ &= \frac{1}{2} \times \frac{1}{3} + 1 \times \frac{1}{3} + 0 \times \frac{1}{3} = \frac{1}{2} \end{aligned}$$

Using the value of  $P(M_{13})$ , students were asked to compute the probabilities of the events  $C_1/M_{13}$  (the event that the car is actually behind door 1 given that Monty opens door 3 when the player chooses door 1) and  $C_2/M_{13}$  (the event that the car is actually behind door 2 given that Monty opens door 3 when the player chooses door 1). The first one yields the probability of winning by sticking to the choice of door 1 and the second yields the probability of winning by switching.

$$P(C_1/M_{13}) = \frac{P(M_{13} \cap C_1)}{P(M_{13})} = \frac{P(M_{13}/C_1)P(C_1)}{P(M_{13})} = \frac{\frac{1}{2} \times \frac{1}{3}}{\frac{1}{2}} = \frac{1}{3}$$

### Students' explorations

At the end of this discussion, some students asked what would happen if there were four doors instead of three. Can the above reasoning be extended to four doors? Many argued that the probability of winning by switching would exceed the probability of winning by sticking (as in the three door case) although they were not sure about how the probabilities would change. One group of students tried to make a tree diagram (reproduced by the author) as shown in Figure 8. This helped them to compute the probabilities of winning by switching and sticking by listing out the elements of the sample space. Another group of students tried to modify their Excel macro to simulate the four door problem. Yet another group of students tried to play the game online using the link [http://onlinestatbook.com/2/probability/monty\\_hall\\_demo.html](http://onlinestatbook.com/2/probability/monty_hall_demo.html) and then arrive at a conclusion. It was interesting to observe students within the same classroom used diverse approaches to analyse the problem. Access to MS Excel and online sources helped to make discussions more meaningful and insightful. Many students conjectured that if there were five doors then the probabilities of winning by sticking and switching would be  $\frac{1}{5}$  and  $\frac{4}{5}$  respectively. The high point of the class was when many students observed that the *probability of winning by switching increases as the number of doors increase*. It was indeed very satisfying for the author when students tried to generalise the problem to  $n$  doors and conjectured that the probabilities of winning by sticking and switching should be  $\frac{1}{n}$  and  $(n-1)/n$  respectively.

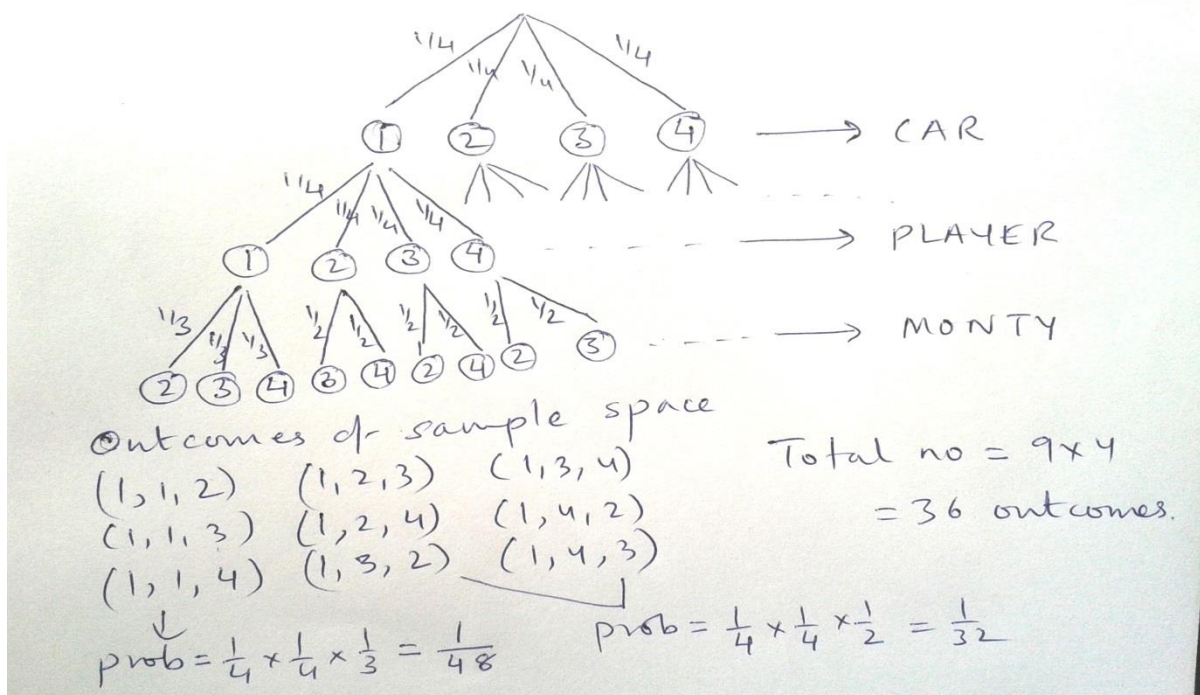


Figure 8: students extended the Monty Hall problem to the case of four doors

### 3.4. Module 3: Newton’s Method in the Complex Plane

The 30 grade 12 students who were a part of the study went through this module. They had been taught differential calculus in their regular class which included the concepts of the limits, continuity and derivatives. They were familiar with the sum, product, quotient and chain rules of differentiation. These students had completed only 4 months of their academic year (in grade 12) and had not been introduced to integral calculus yet. They were also familiar with the concept of fractals through a module on geometric sequences which they had undergone in the previous year (in grade 11). In this module they had explored various attributes of geometrical fractals such as the Sierpinski triangle and Koch snowflake. Refer to Ghosh [5] for an extended discussion on how these students explored fractals. The students had also studied the basic properties of complex numbers represented in the standard as well as polar form. With all this prerequisite knowledge, the author decided to introduce the students to the Newton Raphson method of approximating zeros of polynomials. The module began with using Newton’s method to find the zero of a differentiable function  $f(x)$ .

Given a differentiable function  $f(x)$ , for which we need to find a zero, a sequence of points  $\{x_n\}$  may be generated using the rule

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, 3, \dots$$

Starting with an appropriate initial guess  $x_0$ , the points  $\{x_n\}$  converge to a zero of the function  $f(x)$ . The author supplemented this definition with a pictorial representation explaining how starting from an initial “seed”  $x_0$ , a new point  $x_1$  may be determined by drawing a tangent line to the point  $(x_0, f(x_0))$  on the graph of  $f$ , and following this tangent line from this point to where it intersects the  $x$ -axis.

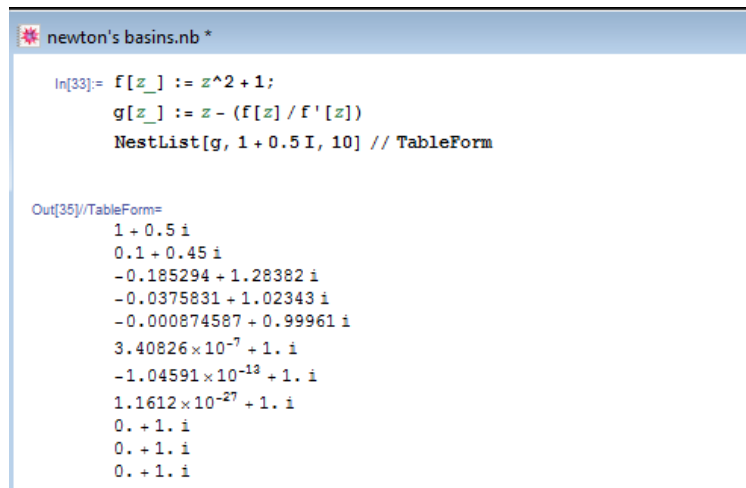
Further, the function  $N(x) = x - \frac{f(x)}{f'(x)}$  was introduced as the *Newton’s function* which is not defined when  $f'(x) = 0$ . It was shown that  $x_1 = N(x_0)$ ,  $x_2 = N(x_1) = N^2(x_0)$ . In general,  $x_n = N^n(x_0)$ .

Students solved exercises where they computed the Newton's sequence for different functions with different initial points  $x_0$ . They were also introduced to the concept of *basins of attraction*.

If  $x^*$  is a zero of  $f$ , the *basin of attraction* of  $x^*$ , is the set of all numbers  $x_0$  such that the Newton sequence starting at  $x_0$  converges to  $x^*$ . This is referred to as the basin of attraction of  $x^*$  and is denoted by  $B(x^*)$ .

After going through the preliminary concepts, students were urged to explore these ideas in the Complex Plane. This transition took place naturally since the derivatives of functions of a complex variable behave computationally just like derivatives of functions of a real variable. However, since conformal maps are not a part of the curriculum, the author focused on a visual approach to helping students understand the concepts. This was achieved by giving students access to Mathematica as they were familiar with the some aspects of Mathematica programming from an earlier module.

To begin with students explored Newton's method in the complex plane by applying the method to the functions  $f(z) = z^2 + 1$  (which has two roots,  $i$  and  $-i$ ) and  $f(z) = z^3 - z$  (which has three roots;  $-1, 0$  and  $1$ ). Students used the NestList command in Mathematica (see Figure 9) to generate the Newton's sequence  $z_1, z_2, z_3, \dots, z_n, \dots$  for these functions. It was interesting to note that all starting points  $z_0$  led to a sequence which converged to some zero of the function. For example, for  $f(z) = z^2 + 1$ , all initial points  $z_0$  either converge to the root  $i$  or to the root  $-i$ .



```

newton's basins.nb *

In[33]:= f[z_] := z^2 + 1;
g[z_] := z - (f[z] / f'[z])
NestList[g, 1 + 0.5 I, 10] // TableForm

Out[35]/TableForm=
  1 + 0.5 i
  0.1 + 0.45 i
 -0.185294 + 1.28382 i
 -0.0375831 + 1.02343 i
 -0.000874587 + 0.99961 i
  3.40826 × 10-7 + 1. i
 -1.04591 × 10-13 + 1. i
  1.1612 × 10-27 + 1. i
  0. + 1. i
  0. + 1. i
  0. + 1. i
  
```

Figure 9: Mathematica code for generating the Newton's iterates for different initial points in the complex plane

### Students' explorations

Students used Mathematica (see Figure 10) to plot the basins of attraction;  $B(i)$  and  $B(-i)$  for  $f(z) = z^2 + 1$ . These turned out to be the half planes created by the line which is the perpendicular bisector of the line segment joining the zeros  $i$  and  $-i$  in the complex plane. This observation helped to confirm Cayley's theorem which is as follows:

Let the complex quadrilateral  $f(z)$  have distinct roots  $\alpha$  and  $\beta$  in the complex plane. Let  $L$  be the perpendicular bisector of the line segment joining the roots  $\alpha$  and  $\beta$ . Then, when Newton's method is applied to  $f(z)$ , the basins of attraction  $B(\alpha)$  and  $B(\beta)$  are exactly the half planes into which  $L$  divides the complex plane.

```

In[4]:= newton[z_] := z - f[z] / f'[z]
plot[r_] := ListDensityPlot[Arg@FixedPoint[newton, #, 50] & /@
  Table[i + j I, {j, -r, r, 2 r/365.}, {i, -r, r, 2 r/365.}],
  ColorFunction -> Hue,
  DataRange -> {{-r, r}, {-r, r}}]
f[z_] := z^4 - 1; plot[2.0];
    
```

Figure 10: Mathematica code for plotting the Newton's basins for the function  $f(z) = z^n - 1$

For  $f(z) = z^3 - 1$ , the above Mathematica code helped to see that Newton's iterates converge to the cube roots of unity. Figure 11 shows the three basins of attraction  $B(1)$ ,  $B(\omega)$ ,  $B(\omega^2)$  (in red, green and purple) as generated by Mathematica. The code assigns a unique color to all the initial points  $z_0$  which converge to a particular root. The graphic thus generated shows an infinite loopy structure at the boundaries of the three basins. By modifying the code, students were able to enlarge a bulb and see the self-similar structure.

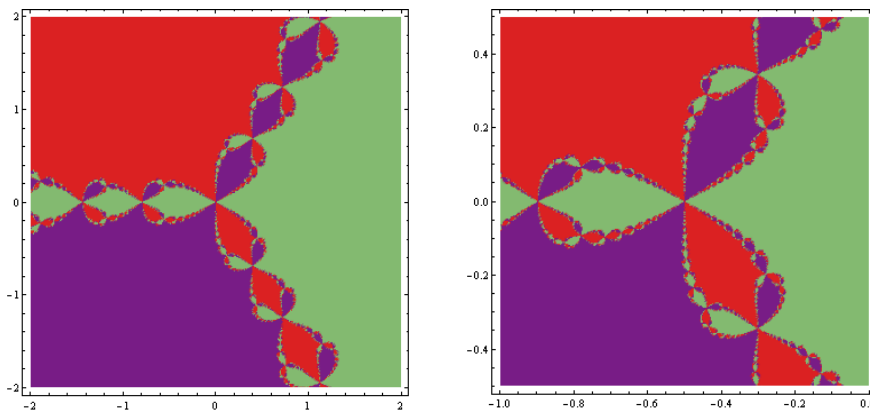


Figure 11: Newton's basins for cube roots of unity

A natural curiosity which arose in the minds of students was to explore the function  $f(z) = z^n - 1$  for various values of  $n$ . Thus, for  $f(z) = z^4 - 1$ , its Newton's iterates converge to the fourth roots of unity,  $1, -1, i, -i$ . The basins of attraction (see Figure 12), visible in different colours, show an interesting structure of four loopy branches formed at the boundaries of the basins.

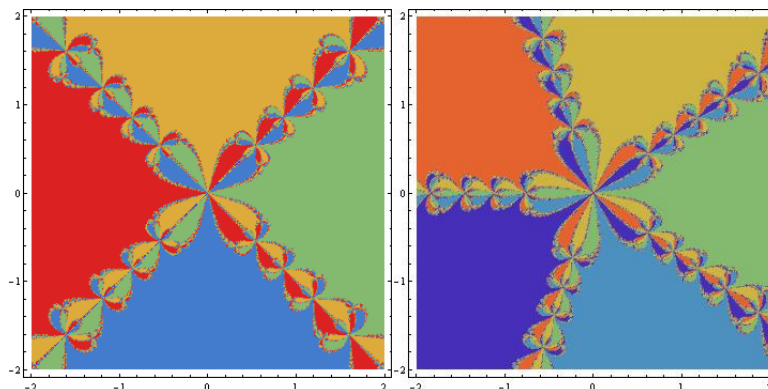


Figure 12: basins of attraction for the functions  $f(z) = z^4 - 1$  and  $f(z) = z^5 - 1$

Students' comments in the feedback session at the end of this module were very encouraging.

*“We were familiar with Newton’s method only as a method for approximating zeros of polynomials in the real line. In this module we learnt how to compute iterates of Newton’s function in the complex plane. The real beauty of all this became clear only when we used Mathematica.”*

*“Mathematica made us see the beautiful patterns which emerged by plotting the basins of attraction of the zeros of different polynomials.”*

*“We were also able to see the fractal patterns within the basins of attraction. All this would not have been possible without Mathematica.”*

#### **4. Conclusions**

In all the modules described in this paper, student’s explorations were facilitated by the author while Mathematica and MS Excel were used as the primary vehicles of exploration. In some cases technology helped to trivialize lengthy calculations while in others it generated powerful graphics which helped to gain a deeper insight into a mathematical concept or process. In the Hill Cipher module, lengthy calculations such as multiplying matrices of large orders and reducing matrices modulo 29, was ‘outsourced’ to Excel. This enabled students to focus on exploring the matrix models and interpreting their solutions. In this module, paper and pencil computations assumed greater importance as in many cases (such as while working with  $2 \times 2$  and  $3 \times 3$  matrices) students preferred to do the calculations by hand and then verify their answers using Excel. In the Monty Hall Problem exploration, Excel helped to simulate the problem quickly thus enabling students to make conjectures regarding the probabilities of winning by ‘switching’ or ‘staying’. It also motivated students to generalize the problem to the case of  $n$  doors. While exploring Newton’s method in the complex plane, the powerful graphics features of Mathematica helped to visualize the iterates of various functions by plotting the basins of attraction of different roots. Students were able to visualize the rich nature of Newton’s basins and also observe fractal structures within them.

Throughout the modules, technology provided ample opportunity for visualization and exploration. It also served as a ‘mathematical investigation assistant’, as proposed by Arnold [1], giving students control over what they were learning. While exploring the applications of the various topics, students were actively engaged in processes such as estimation, approximation, making conjectures and generalizing patterns – a point strongly recommended by the National Curriculum Framework (NCF) 2005. Finally, the most important outcome of the modules was that technology enabled explorations of the mathematical applications gave students access to higher level mathematical concepts. This is particularly evident from the fact that modular linear algebra (required to understand the Hill Cipher method and the Cracking Theorem) and Newton’s basins in the complex plane are much beyond the level of school mathematics. According to Heid [7], the technology acted as an ‘amplifier’ as students learnt new mathematical content, some of which was far beyond the level of grade 12.

To summarise, it may be apt to say that integration of appropriate mathematical applications aided by facilitation by the teacher, enabled the students to strengthen their basic mathematical content knowledge and also gave them access to new concepts and ideas which were beyond the scope of the curriculum. Further, the considered use of technology helped to strike a balance between computational skills and processes such as visualization and exploration.



## 5. References

- [1] Arnold, S., *Classroom computer algebra: some issues and approaches*, Australian Mathematics Teacher, 60(2), 17-21, 2004.
- [2] Eisenberg, M., *Hill Ciphers and Modular Linear Algebra*, 1999, Retrieved from <http://www.apprendre-en-ligne.net/crypto/hill/Hillciph.pdf>,
- [3] Ghosh, J., *The Monty Hall problem: A spreadsheet simulation*. At Right Angles, 1 (2), 57-62., 2012.
- [4] Ghosh, J., *Technology in the mathematics classroom: A perspective from India*. Proceedings of the 20<sup>th</sup> Asian Technology Conference in Mathematics, 110-124., 2015.
- [5] Ghosh, J., *Algebraic thinking through Koch snowflake constructions*. Mathematics Teacher, 109(9), 693-99., 2016.
- [6] Heid, M.,K. *Resequencing skills and concepts in applied calculus using the computer as a tool*. Journal for Research in Mathematics Education, 19(1), 3-25, 1988.
- [7] Heid, M.,K. *Theories that inform the use of CAS in the teaching and learning of mathematics*. Plenary paper presented at the Computer Algebra in Mathematics Education (CAME) 2001 symposium, Retrieved from <http://www.lkl.ac.uk/research/came/events/freudenthal/3-Presentation-Heid.pdf>
- [8] Kissane, B., *Some calculus affordances of a graphics calculator*. Australian Senior Mathematics Journal, 22(2), 15-27, 2008.
- [9] Laborde, C., *Dynamic geometry environments as a source for rich learning contexts for the complex activity of proving*. Educational Studies in Mathematics, 44(1), 151-161, 2000.
- [10] Leung, A., *Discernment and reasoning in dynamic geometry environments*. Regular lecture delivered at 12th International Congress on Mathematical Education, Seoul, Korea, 2012.
- [11] Stacey, K., *Teaching with CAS in a time of transition*. Plenary paper presented at the Computer Algebra in Mathematics Education (CAME) 2001 symposium. Retrieved from <http://www.lkl.ac.uk/research/came/events/freudenthal/2-Presentation-Stacey.pdf>
- [12] National Council for Educational Research and Training. Position paper of National Focus Group on Teaching of Mathematics, 2005. Retrieved from <http://www.ncert.nic.in/rightside/links/pdf/framework/nf2005.pdf>